

SEMIFLAT L -CLUSTER METHODS

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In previous work, an order theoretic model for cluster analysis has been developed, and flat cluster methods characterized in terms of their compatibility with respect to the set of residuated mappings on the non-negative reals. This suggests the possibility of classifying cluster methods according to their compatibility properties with respect to residuated mappings. Such a program is herewith initiated. Semiflat cluster methods are defined and characterized by their compatibility with respect to those residuated mappings θ for which $\theta^+(0) = 0$. It is also shown how these methods fit into an earlier graph theoretic model for cluster analysis that was developed by Jardine and Sibson.

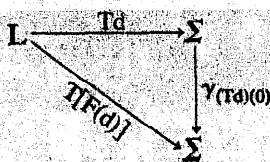
In [1] we introduced a model for cluster analysis — a model that involves partially ordered sets. Our characterization of flat L -cluster methods [1, Theorems 6.2 and 7.4] suggests that it might be profitable to classify L -cluster methods according to their compatibility with respect to certain residuated mappings on the underlying join semilattice L in which all dissimilarities are measured. In this paper, we shall start such a project. In that the basic notation and terminology will follow that of [1], we shall not specifically reintroduce it here. However, in the interest of readability, we *shall* take the time to mention that P denotes a finite set of objects to be classified (with $|P| = p \geq 2$), PP denotes the set of 2 element subsets of P , Σ the Boolean algebra of all subsets of PP , and L a fixed join semilattice with 0 such that $|L| \geq 2$. An L -cluster method is a mapping $F: LC(P) \rightarrow LC(P)$, where $LC(P)$ denotes the set of all mappings $d: PP \rightarrow L$. These play the role of dissimilarity coefficients on P (see [2, p. 77]). Alternately, an L -cluster method may be viewed as a mapping $\bar{F}: \text{Res}^+(L, \Sigma) \rightarrow \text{Res}^+(L, \Sigma)$. Here $\text{Res}^+(L, \Sigma)$ denotes the set of residual mappings of L into Σ . Finally, for $d \in LC(P)$, $Td \in \text{Res}^+(L, \Sigma)$ is defined by the requirement that for each $h \in L$,

$$(Td)(h) = \{ \{a, b\} : d(a, b) \leq h \}.$$

We are now ready to introduce the concept of a semiflat L -cluster method.

Definition 1.1. An L -cluster method F is said to be *semiflat* if there is a family $(\gamma_s)_{s \in \Sigma}$ of mappings of Σ such that $T[F(d)] = \gamma_{(Td(0))} \circ Td$ for all $d \in LC(P)$.

In other words, to say that an L -cluster method F is semiflat is to say that there is a family of commutative diagrams of the following form:



As in the case of a flat L -cluster method [1, §5], this leads to a pair of problems.

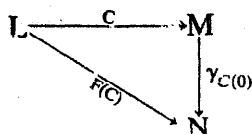
First semiflat cluster problem. Given a family $(\gamma_s)_{s \in \Sigma}$ of mappings of Σ , find necessary and sufficient conditions for there to exist a semiflat L -cluster method F such that $T[F(d)] = \gamma_{T(d)(0)} \circ Td$ for all $d \in LC(P)$.

Second semiflat cluster problem. Suppose the L -cluster method F is given. What properties must F have in order that there exist a family $(\gamma_s)_{s \in \Sigma}$ of mappings of Σ so that $T[F(d)] = \gamma_{T(d)(0)} \circ Td$ for every $d \in LC(P)$?

In the next two sections, these problems will be considered separately. Following this, a section will be devoted to the characterization of semiflat cluster methods within the context of the model for cluster analysis proposed by Jardine and Sibson in [2]. In closing this section, we make an assumption about L . If $|L| = 2$, a moments reflection should convince you that every L -cluster method is semiflat. For that reason, we now specifically assume that for the remainder of the paper, $|L| > 2$.

2. First semiflat cluster problem

As in our discussion of flat L -cluster methods [1, §6] it will cost very little to phrase the problem in an abstract setting. Accordingly, we let M, N denote partially ordered sets with 1, neither of which is a chain. We consider a family $(\gamma_m)_{m \in M}$ of mappings of M into N , and agree to call the family L -semiflat in case there is a function $F: \text{Res}^+(L, M) \rightarrow \text{Res}^+(L, N)$ completing the following diagram:



This amounts to the assertion that $C \in \text{Res}^+(L, M)$ should imply that $\gamma_{C(0)} \circ C \in \text{Res}^+(L, N)$. The first semiflat cluster problem now takes the following form:

Find necessary and sufficient conditions for a given family of mappings of M into N to be L -semiflat.

In connection with this problem, we observe that for $x \neq C(0)$, the value of $\gamma_{C(0)}(x)$ has no effect on whether $\gamma_{C(0)} \circ C$ is residual. In order to avoid this arbitrary

feature, we agree to restrict our attention to families $(\gamma_m)_{m \in M}$ having the property that for $x \not\geq m$, $\gamma_m(x) = \gamma_m(m)$. We then have

Lemma 2.1. *Let $(\gamma_m)_{m \in M}$ be an L -semiflat family. Then:*

- (1) $\gamma_m(1_M) = 1_N$ for all $m \in M$.
- (2) $\gamma_{1_M}(m) = 1_N$ for all $m \in M$
- (3) $m \leq m'$ in M implies $\gamma_m(m) \leq \gamma_m(m')$ in N .

Proof. (1) Choose $C \in \text{Res}^+(L, M)$ so that $C(0) = m$. There must then exist $h_1, h_2 \in L$ with $C(h_1) = 1_M$ and $(\gamma_m \circ C)(h_2) = 1_N$. Setting $h = h_1 \vee h_2$: we have

$$\gamma_m(1_M) = \gamma_m(C(h)) = (\gamma_m \circ C)(h) = 1_N$$

(2) By (1), $\gamma_{1_M}(1_M) = 1_N$, and by hypothesis, $\gamma_{1_M}(x) = \gamma_{1_M}(1_M) = 1_N$ for $x \not\geq 1_M$.

(3) Let $m < m'$ in M , and choose elements $h, k \in L$ so that $0 < h < k$. As in the proof of [1, Lemma 6.1], we may find $C \in \text{Res}^+(L, M)$ so that $C(0) = m$ and $C(h) = m'$. Then

$$\gamma_m(m) = (\gamma_m \circ C)(0) \leq (\gamma_m \circ C)(h) = \gamma_m(m'),$$

as desired.

In view of the above lemma, we agree to call a family $(\gamma_m)_{m \in M}$ of mappings of M into N *potentially L -semiflat* in case:

- (1) $\gamma_m(x) = \gamma_m(m)$ for all $x \not\geq m$;
- (2) $\gamma_m(1_M) = 1_N$ for all $m \in M$;
- (3) $\gamma_{1_M}(m) = 1_N$ for all $m \in M$;
- (4) $\gamma_m(m) \leq \gamma_m(m')$ if $m \leq m'$ in M .

The content of Lemma 2.1 is that *every L -semiflat family is potentially L -semiflat*. As in the case of L -flat mappings [1, Theorem 6.2], the exact nature of L -semiflat families of mappings depends on the structure of L . We proceed with the analogue of [1, Lemma 6.1], and divide the results up into a pair of lemmas.

Lemma 2.2. *Let $(\gamma_m)_{m \in M}$ be an L -semiflat family of mappings of M into N . Then:*

- (1) If L has height greater than 3, then each γ_m is isotone.
- (2) Suppose L is not a chain. Then $s \wedge t = m$ in M implies that $\gamma_m(m)$ is the meet in N of $\{\gamma_m(s), \gamma_m(t)\}$.
- (3) Assume L has a nonzero element h_0 such that $\{k \in L : k \geq h_0\}$ is not a chain. Then each γ_m is a meet homomorphism.

Proof. (1) Let $s < t$ in M . If $m \not\leq s$, then $\gamma_m(s) = \gamma_m(m) \leq \gamma_m(t)$, and if $t = 1_M$ we may apply Lemma 2.1 to deduce that $\gamma_m(s) \leq \gamma_m(t)$. This leaves us with the case

where $m < s < t < 1_M$. Using the fact that L has height greater than 3, we choose a chain $0 < h < k < h_1$ of elements of L , and define $C: L \rightarrow M$ by the rule

$$C(y) = \begin{cases} 1_M, & y \geq h_1, \\ t, & y \geq k, y \not\geq h_1, \\ s, & y \geq h, y \not\geq k, \\ m, & y \not\geq h. \end{cases}$$

Noting that $C \in \text{Res}^+(L, M)$, it is now clear that

$$\gamma_m(s) = (\gamma_m \circ C)(h) \leq (\gamma_m \circ C)(k) = \gamma_m(t).$$

(2) Since L is not a chain, we may choose $h, k \in L$ that are not comparable; furthermore, we may as well assume that s, t are not comparable. As in the proof of [1, Lemma 6.1(5)], we now choose $C \in \text{Res}^+(L, M)$ so that

$$C(y) = \begin{cases} 1_M, & y \geq h \vee k, \\ s, & y \geq h, y \not\geq k, \\ t, & y \geq k, y \not\geq h, \\ m, & y \not\geq h, y \not\geq k. \end{cases}$$

Proceeding as in the proof of [1, Lemma 6.1(5)], the reader can now verify that $\gamma_m(m)$ is the meet in N of $\{\gamma_m(s), \gamma_m(t)\}$.

(3) Let $s, t \in M$, and assume that $s \wedge t$ exists. We must show that for arbitrary $m \in M$, $\gamma_m(s \wedge t)$ is the meet in N of $\{\gamma_m(s), \gamma_m(t)\}$. If $m \not\leq s$ or $m \not\leq t$, then $\gamma_m(s \wedge t) = \gamma_m(m) = \gamma_m(s) \wedge \gamma_m(t)$, and there is nothing to prove. Thus we may as well assume that $m \leq s \wedge t$. If $s \wedge t = m$, the desired result follows from (2), so we even assume that $m < s \wedge t$. By our assumption on L , we may now choose $h, k \in L$ so that $0 < h_0 < h$, $0 < h_0 < k$, and h, k are not comparable. Then L has a chain of the form $0 < h_0 < h < h \vee k$, so by (1), each γ_m is isotone. For that reason, we assume that s, t are not comparable. At this point, we have $m < s \wedge t < s$ and $m < s \wedge t < t$. We now define $C: L \rightarrow M$, $C^*: M \rightarrow L$ by the rules

$$C(y) = \begin{cases} 1_M, & y \geq h \vee k, \\ s, & y \geq h, y \not\geq k, \\ t, & y \geq k, y \not\geq h, \\ s \wedge t, & y \geq h_0, y \not\geq h, y \not\geq k, \\ m, & y \not\geq h_0; \end{cases}$$

$$C^*(x) = \begin{cases} 0, & x \leq m, \\ h_0, & x \leq s \wedge t, x \not\leq m, \\ h, & x \leq s, x \not\leq t, \\ k, & x \leq t, x \not\leq s, \\ h \vee k, & x \not\leq s, x \not\leq t. \end{cases}$$

We leave to the reader the routine verification that C^* is residuated with C its associated residual mapping. Since $C(0) = m$, we know that $\gamma_m \circ C \in \text{Res}^+(L, N)$.

An argument similar to that given in [1, Lemma 6.1(5)] will now show that $\gamma_m(s \wedge t)$ is effective as the meet in N of $\{\gamma_m(s), \gamma_m(t)\}$.

Let $(\gamma_m)_{m \in M}$ be a potentially L -semiflat family of mappings on M into N . It will turn out to be useful to say that γ_m is a *weak meet homomorphism* in case $s \wedge t = m$ in M implies that $\gamma_m(m)$ is the meet in N of $\{\gamma_m(s), \gamma_m(t)\}$. With this in mind, we now state.

Lemma 2.3. *Let $(\gamma_m)_{m \in M}$ be a potentially L -semiflat family of mappings of M into N . Each of the following conditions is sufficient for $(\gamma_m)_{m \in M}$ to be L -semiflat:*

- (1) L is a 3 element chain.
- (2) L is a chain with more than 3 elements, M is finite, and each γ_m is isotone.
- (3) L is not a chain, but L does have height 3; furthermore, each γ_m is a weak meet homomorphism.
- (4) L is not a chain, L has height greater than 3, and for each nonzero element $h \in L$, $\{k \in L : k \geq h\}$ is a chain; furthermore, M is finite, and each γ_m is an isotone weak homomorphism.

Proof. (1) Let $L = \{0, h, 1\}$. Choose $C \in \text{Res}^+(L, M)$, and set $r = C(0)$, $s = C(h)$. Consider a typical $n \in N$. If $n \not\leq \gamma_r(s)$, then $(\gamma_r \circ C)(k) \geq n \Leftrightarrow k = 1$; if $n \leq \gamma_r(s)$, but $n \not\leq \gamma_r(r)$, then $(\gamma_r \circ C)(k) \geq n \Leftrightarrow k \geq h$; if $n \leq \gamma_r(r)$, then $(\gamma_r \circ C)(k) \geq n$ for all $k \in L$. Hence $\gamma_r \circ C \in \text{Res}^+(L, N)$.

(2) The proof proceeds in a manner similar to that of [1, Lemma 6.1(4)].

(3) Let $C \in \text{Res}^+(L, M)$ with $r = C(0)$. For a given $n \in N$, we consider $\{k \in L : (\gamma_r \circ C)(k) \geq n\}$. Since L must have a largest element 1, this set is nonempty; indeed, $(\gamma_r \circ C)(1) = \gamma_r(1_M) = 1_N \geq n$. If 1 is its only member, then $(\gamma_r \circ C)(k) \geq n \Leftrightarrow k = 1$. If there is a unique y ($0 < y < 1$) in the set, then $(\gamma_r \circ C)(k) \geq n \Leftrightarrow k \geq y$. If 0 is in the set, then $\gamma_r(r) = (\gamma_r \circ C)(0) \geq n$ shows that $(\gamma_r \circ C)(k) \geq n$ for every $k \in L$. The only other possibility is for y_1, y_2 to be in the set with $0 < y_1 < 1, 0 < y_2 < 1$. By our assumption on the height of L , this forces $y_1 \wedge y_2 = 0$, and since C is residual, we have $r = C(0) = C(y_1 \wedge y_2) = C(y_1) \wedge C(y_2)$. Hence $\gamma_r(r) = (\gamma_r \circ C)(y_1) \wedge (\gamma_r \circ C)(y_2) \geq n$ shows that 0 is in the set. Thus in all cases, we can find an element $y \in L$ such that $(\gamma_r \circ C)(k) \geq n \Leftrightarrow k \geq y$. But this says that $\gamma_r \circ C \in \text{Res}^+(L, N)$.

(4) Let $C \in \text{Res}^+(L, M)$ with $C(0) = r$. For $n \in N$, we consider $\{m \in M : \gamma_r(m) \geq n\}$ and let m_1, \dots, m_q be the minimal elements of this set. Then choose $k_1, \dots, k_q \in L$ so that $C(k_i) \geq m_i \Leftrightarrow k_i \geq k_i$. Suppose k_i, k_j are not comparable in L . Then $k_i \wedge k_j = 0$, and

$$\begin{aligned} \gamma_r(r) &= (\gamma_r \circ C)(0) = (\gamma_r \circ C)(k_i \wedge k_j) \\ &= (\gamma_r \circ C)(k_i) \wedge (\gamma_r \circ C)(k_j) \geq n, \end{aligned}$$

whence $(\gamma_r \circ C)(k) \geq n$ for all $k \in L$. On the other hand, if k_i, k_j are comparable for

every i, j , then we may choose k to be the minimum of $\{k_1, \dots, k_q\}$ and note that

$$(\gamma_r \circ C)(h) \geq n \Leftrightarrow C(h) \geq \text{some } m_i \Leftrightarrow h \geq \text{some } k_i \Leftrightarrow h \geq k.$$

We deduce that $\gamma_r \circ C \in \text{Res}^+(L, N)$.

At this point, we pause to consider some conditions on a family $G = (\gamma_m)_{m \in M}$ of mappings of M into N :

- (G1) G is potentially L -semi-flat.
- (G2) G is a potentially L -semi-flat family of isotone mappings.
- (G3) G is a potentially I -semi-flat family of weak meet homomorphisms.
- (G4) G is a potentially L -semi-flat family of isotone weak meet homomorphisms.
- (G5) G is a potentially L -semi-flat family of meet homomorphisms.

Evidently, $(G5) \Rightarrow (G4) \xRightarrow{(G3)} (G1)$
 $\xRightarrow{(G2)}$

We shall call the pair (M, N) a *rich pair* of partially ordered sets if the above conditions are all distinct. Though we shall not attempt a characterization of rich pairs of partially ordered sets, we do mention that if M, N are partially ordered sets with 1, then the following conditions will guarantee that (M, N) be a rich pair:

(R1) N has a pair of noncomparable elements having a common lower bound.

(R2) M has a sub-partially ordered set whose Hasse diagram appears in Fig. 1.

By this we mean that the indicated meets actually occur in M .

We now have the analogue of [1, Theorem 6.2].

Theorem 2.4. Let M, N be partially ordered sets with 1, neither of which is a chain. Assume further that the pair (M, N) is rich, and that $G = (\gamma_m)_{m \in M}$ denotes a family of mappings of M into N . Then $(1a) \Leftrightarrow (1b)$. If M is finite, the implications $(2a) \Leftrightarrow (2b)$, $(3a) \Leftrightarrow (3b)$, and $(4a) \Leftrightarrow (4b)$ also hold, and if M, N are finite lattices, $(5a) \Leftrightarrow (5b)$.

(1a) L is a 3 element chain.

(1b) G is L -semi-flat iff it is potentially L -semi-flat.

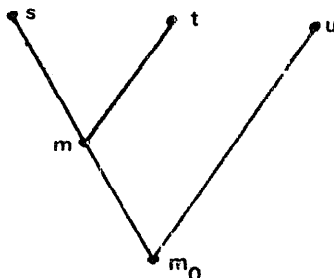


Fig. 1.

(2a) L is a chain with more than 3 members.

(2b) G is L -semiflat iff it is a potentially L -semiflat family of isotone mappings.

(3a) L has height 3, but is not a chain.

(3b) G is L -semiflat iff it is a potentially L -semiflat family of weak meet homomorphisms.

(4a) L has height greater than 3, L is not a chain, and for each $h > 0$, $\{k \in L : k \geq h\}$ is a chain.

(4b) G is L -semiflat iff it is a potentially L -semiflat family of isotone weak meet homomorphisms.

(5a) L has a nonzero element h such that $\{k \in L : k \geq h\}$ is not a chain.

(5b) G is L -semiflat iff it is a potentially L -semiflat family of residual mappings.

Proof. (1a) \Rightarrow (1b) By Lemma 2.3(1), every potentially L -semiflat family G is L -semiflat. The converse is the content of Lemma 2.1.

(1b) \Rightarrow (1a) Choose G so that some γ_m is not isotone, and so some γ_r is not a weak meet homomorphism. By Lemma 2.2(2), L must be a chain, and by Lemma 2.2(1), it must be a 3 element chain.

(2a) \Rightarrow (2b) By Lemma 2.2(1), every L -semiflat family is a potentially L -semiflat family of isotone mappings. The converse follows from Lemma 2.3(2).

(2b) \Rightarrow (2a) Choose G to be an L -semiflat family so that each γ_m is isotone, but so that some γ_r is not a weak meet homomorphism. By Lemma 2.2(2), L must be a chain, and by (1a) \Leftrightarrow (1b), L must have more than 3 members.

(3a) \Rightarrow (3b) Lemmas 2.2(2) and 2.3(3).

(3b) \Rightarrow (3a) Take G to be an L -semiflat family of weak meet homomorphisms such that some γ_m is not isotone. By Lemma 2.2(1), L must have height 3, and by (1a) \Leftrightarrow (1b), it cannot be a chain.

(4a) \Rightarrow (4b) Lemmas 2.2(1), 2.2(2), and 2.3(4).

(4b) \Rightarrow (4a) By (1a) \Leftrightarrow (1b) and (2a) \Leftrightarrow (2b), L cannot be a chain, and by (3a) \Leftrightarrow (3b), L must have height greater than 3. Choose G to be an L -semiflat family of isotone weak meet homomorphisms such that some γ_m is not a meet homomorphism. By Lemma 2.2(3), for each $h > 0$, $\{k \in L : k \geq h\}$ is a chain.

(5a) \Rightarrow (5b) In view of the fact that M, N are assumed to be finite lattices, a mapping $\gamma : M \rightarrow N$ is residual iff it is a meet homomorphism and maps 1_M into 1_N . In view of this, Lemma 2.2(3) says that every L -semiflat family is a family of residual mappings. The converse follows from the fact that the composition of residual mappings is residual.

(5b) \Rightarrow (5a) This follows from the equivalences we have already established.

It is time to apply all of this back in the context in which it originated. We therefore return to the study of L -cluster methods. We take P to be a finite set, and $M = N = \Sigma$, the Boolean algebra of all subsets of PP , the set of 2 element subsets of P . In order to be certain that (Σ, Σ) be rich, we require that P have at least 3 members. Then Σ satisfies both (R1) and (R2), so the requirements of the above theorem are all satisfied, and we have all of the equivalences that it asserts.

3. Second semiflat cluster problem

As in our treatment of the first semiflat cluster problem, it will cost very little to state the problem in an abstract setting. Thinking back to our solution of the second flat cluster problem [1, §7] we are led to seek a solution that involves compatibility with respect to certain residuated mappings on L . The reader should recall that to say that an L -cluster method F is θ -compatible for $\theta \in \text{Res}(L)$ is to say that $F(\theta d) = \theta F(d)$ for all $d \in LC(P)$.

In view of the above discussion, we now take M, N to be partially ordered sets with 1, and assume that each has more than 1 member. We then consider a mapping $F: \text{Res}(M, L) \rightarrow \text{Res}(N, L)$, and note that F induces a mapping $\bar{F}: \text{Res}^+(L, M) \rightarrow \text{Res}^+(L, N)$ by the rule $\bar{F}C = (FC^*)^+$, where C^* is chosen so that $C^{**} = C$. To say that F is semiflat will be to say that there is a family $(\gamma_m)_{m \in M}$ of mappings of M into N such that $\bar{F}C = \gamma_{C(0)} \circ C$ for all $C \in \text{Res}^+(L, M)$. To say that F is θ -compatible for $\theta \in \text{Res}(L)$ will be to say that $F(\theta \circ C^*) = \theta \circ F(C^*)$ for all $C^* \in \text{Res}(M, L)$. In connection with this, we recall that by [1, Lemma 7.1], $F(\theta \circ C^*) = \theta \circ F(C^*)$ if and only if $\bar{F}(C \circ \theta^+) = (\bar{F}C) \circ \theta^+$. Our problem is to decide when F is semiflat in terms of those residuated mappings θ on L for which F is θ -compatible. To indicate that there is some hope in accomplishing such a project, we begin with

Lemma 3.1. *Every semiflat mapping $F: \text{Res}(M, L) \rightarrow \text{Res}(N, L)$ is θ -compatible for all $\theta \in \text{Res}(L)$ such that $\theta^+(0) = 0$.*

Proof. We simply note that $C(0) = (C \circ \theta^+)(0)$, so that $\bar{F}(C \circ \theta^+) = \gamma_{C(0)} \circ (C \circ \theta^+) = (\gamma_{C(0)} \circ C) \circ \theta^+ = (\bar{F}C) \circ \theta^+$.

Before we can proceed, we shall need some additional examples of residuated mappings on L . These are provided by the following lemma, whose routine proof is omitted.

Lemma 3.2. *Let $0 < h < h_1$, $0 < k < h_1$ in L . Define $\delta: L \rightarrow L$ by*

$$\delta(y) = \begin{cases} 0, & y = 0, \\ k, & 0 < y \leq h, \\ y \vee h_1, & y \not\leq h. \end{cases}$$

Then $\delta \in \text{Res}(L)$ with δ^+ given by

$$\delta^+(y) = \begin{cases} y, & y \geq h_1, \\ h, & y \geq k, y \not\geq h_1, \\ 0, & y \not\geq k. \end{cases}$$

Assume now that $F: \text{Res}(M, L) \rightarrow \text{Res}(N, L)$ has the property that:

(I) If L does not have a largest element, then F is θ -compatible for all $\theta \in \text{Res}(L)$ that are increasing in that $\theta(h) \geq h$ for all $h \in L$.

(II) If L does have a largest element, then F is θ -compatible for all $\theta \in \text{Res}(L)$ such that $\theta^+(0) = 0$.

Our goal is to show that F is semiflat. The proof will be broken up into a series of lemmas. For ease of reference, we shall divide the proof of each of the next three lemmas into two cases. In Case I, we shall assume that L does not have a largest element, and in Case II, that it does. When L does have a largest element 1, it will prove useful to know that for each $h \in L$, the mapping $\alpha_h: L \rightarrow L$ defined by $\alpha_h(0) = 0$, $\alpha_h(k) = h(k \neq 0)$ is residuated with α_h^+ given by $\alpha_h^+(k) = 1(k \geq h)$, $\alpha_h^+(k) = 0(k \not\geq h)$. We define $\bar{0}_M: M \rightarrow L$ by $\bar{0}_M(m) = 0$ for all $m \in M$, and $\bar{0}_N: N \rightarrow L$ in a similar fashion. We are now ready to proceed.

Lemma 3.3. $F(\bar{0}_M) = \bar{0}_N$.

Proof. Case I. Since L does not have a largest element, we may find a chain of the form $0 < h < k < h_1$. By Lemma 3.2, there is an increasing residuated mapping δ on L such that $\delta(h) = k$.

Case II. Here we choose k so that $k > 0$ and $k \neq h$. The mapping α_k has the property that it is residuated, $\alpha_k^+(h) = 0$, and $\alpha_k(h) = k$.

In either case, we may find $\theta \in \text{Res}(L)$ such that F is θ -compatible, and such that $\theta(h) \neq h$. But $F(\bar{0}_M) = F(\theta \circ \bar{0}_M) = \theta \circ F(\bar{0}_M)$ forces $h = \theta(h)$, a contradiction.

Lemma 3.4. Let $h, k > 0$ in L , and $C, D \in \text{Res}^+(L, M)$. Then.

(1) If the range of C^* is $\{0, h\}$, the range of $F(C^*)$ is $\{0\}$ or $\{0, h\}$.

(2) If the range of C^* is $\{0, h\}$, the range of D^* is $\{0, k\}$, and if $C(0) = D(0)$, then $(\bar{F}C)(0) = (\bar{F}D)(0)$.

Proof. (1) Case 1. We choose h_1 so that $h_1 > h \vee (FC^*)(1_N)$, and note that $h < h_1$. If δ is defined by

$$\delta(y) = \begin{cases} 0, & y = 0, \\ h, & 0 < y \leq h, \\ y \vee h_1, & y \not\leq h. \end{cases}$$

then by Lemma 3.2 (with $h = k$), $\delta \in \text{Res}(L)$. Evidently δ is increasing, and the fact that $C^* = \delta \circ C^*$ shows that $F(C^*) = \delta \circ F(C^*)$. Since $(FC^*)(1_N) < h_1$, it follows that the range of $F(C^*)$ is $\{0\}$ or $\{0, h\}$.

Case II. The mapping α_h has the property that $C^* = \alpha_h \circ C^*$, so $F(C^*) = \alpha_h \circ F(C^*)$. In that the range of α_h is $\{0, h\}$, this shows the range of $F(C^*)$ to be either $\{0\}$ or $\{0, h\}$.

(2) Case I. If $h < k$, we may choose $h_1 > k$, and use Lemma 3.2 to produce an increasing residuated mapping δ such that $\delta(h) = k$. Then $D^* = \delta \circ C^*$, so $\bar{F}D = (\bar{F}C) \circ \delta^+$, and $(\bar{F}D)(0) = [(\bar{F}C) \circ \delta^+](0) = (\bar{F}C)(0)$.

If $h \not\leq k$, we choose $h_1 > h \vee k$, and apply the above argument to the pair $h, h \vee k$ as well as to the pair $k, h \vee k$.

Case I'. Here we let $\delta = \alpha_k$, note that $D^* = \delta \circ C^*$, and proceed as in Case I.

Lemma 3.5. Let $0 < h < h_1$, $0 < k < k_1$ in L , and $C, D \in \text{Res}^+(L, M)$. Suppose that the range of C^* is $\{0, h, h_1\}$ and the range of D^* is $\{0, k, k_1\}$. Suppose further that $C(0) = D(0)$ and $C(h) = D(k)$. Then $(\bar{F}C)(h) = (\bar{F}D)(k)$.

Proof. Assume first that $h = k$. If $k_1 > h_1$, we define δ by

$$\delta(y) = \begin{cases} 0, & y = 0, \\ h, & 0 < y \leq h, \\ k \vee y, & y \not\leq h. \end{cases}$$

Then $\delta \in \text{Res}(L)$, δ is increasing, and $D = \delta \circ C$; hence $\bar{F}D = (\bar{F}C) \circ \delta^+$, so $(\bar{F}D)(h) = [(\bar{F}C) \circ \delta^+](h) = (\bar{F}C)(h)$. For the general situation, the above argument is applied to the pair $h_1, h_1 \vee k_1$ and to the pair $k_1, h_1 \vee k_1$.

We now consider the situation where $h \neq k$. By what has just transpired, we may assume that $h_1 = k_1$.

Case I. If $h < k$, we define δ as in Lemma 3.2, and note that $D^* = \delta \circ C^*$, $\delta^+(k) = h$. Hence $\bar{F}D = (\bar{F}C) \circ \delta^+$, and $(\bar{F}D)(k) = [(\bar{F}C) \circ \delta^+](k) = (\bar{F}C)(h)$. In general, this argument is applied to the pair $h, h \vee k$ and to the pair $k, h \vee k$, with h_1 chosen so that $h_1 > h \vee k$.

Case II. As in Case I, we may assume that $h_1 = k_1$, and use Lemma 3.2 to define δ so that $\delta \in \text{Res}(L)$, $\delta^+(0) = 0$, $\delta(h) = k$, and $\delta^+(k) = h$. We then proceed as in Case I.

Lemma 3.6. Let $C \in \text{Res}^+(I, M)$. If $C(h) = 1_M$, then $(\bar{F}C)(h) = 1_N$.

Proof. If h is the largest element of L , the assertion is clear. If not, there is an $h_1 > h$; indeed, h_1 may be chosen so that $h_1 > h$ and $h_1 \geq (\bar{F}C)(1_N)$. We now define δ by

$$\delta(y) = \begin{cases} 0, & y = 0, \\ h, & 0 < y \leq h, \\ y \vee h_1, & y \not\leq h_1. \end{cases}$$

Then $\delta \in \text{Res}(L)$, δ is increasing, and the range of $D^* = \delta \circ C^*$ is $\{0, h\}$. By Lemma 3.4, the range of $F(D^*)$ is $\{0\}$ or $\{0, h\}$. Hence $(\bar{F}D)(h) = 1_N$. Since $\delta^+(h) = h$, it follows that $(\bar{F}C)(h) = [(\bar{F}C) \circ \delta^+](h) = (\bar{F}D)(h) = 1_N$.

At this point we are ready to define our mappings $(\gamma_m)_{m \in M}$ of M into N . We begin by defining $\gamma_{1_M}(r) = 1_N$ for all $r \in M$. Suppose $m \in M$ and $m \neq 1_M$. For $h > 0$, we choose $C \in \text{Res}^+(L, M)$ so that the range of C^* is $\{0, h\}$ and so that $C(0) = m$. We then define $\gamma_m(m) = (\bar{F}C)(0)$, and note that by Lemma 3.4, this definition is independent of the choice of h . For $m < r < 1_M$, we take $0 < h < h_1$ in L , and

choose $C \in \text{Res}^+(L, M)$ so that the range of C^* is $\{0, h, h_1\}$, $C(0) = m$, and $C(h) = r$. We then let $\gamma_m(r) = (\bar{F}C)(h)$, and note that by Lemma 3.5, this definition is independent of the choice of h and h_1 . Finally, γ_m is extended to all of M by defining $\gamma_m(1_M) = 1_N$, and $\gamma_m(r) = \gamma_m(m)$ if $r \not\geq m$.

Lemma 3.7. For $C \in \text{Res}^+(L, M)$, $\bar{F}(C) = \gamma_{C(0)} \circ C$.

Proof. First of all, since $F(\bar{0}_M) = \bar{0}_N$, we have for all $h \in L$ that $[\bar{F}(\bar{0}_M^+)](h) = \bar{0}_N^+(h) = 1_N = (\gamma_{1_M} \circ \bar{0}_M)(h)$, so the result is true for $C = \bar{0}_M^+$. We now choose $C \in \text{Res}^+(L, M)$ and assume that $C \neq \bar{0}_M^+$. Let $m = C(0)$ and for a given $h > 0$, $r = C(h)$. We now set $h_1 = C^*(1_M) \vee (FC^*)(1_N)$ and consider the following cases:

Case 1. $r = 1_M$. Then by Lemma 3.6 $(\bar{F}C)(h) = 1_N = \gamma_m(r)$.

Case 2. $r = m$. We claim that $(\bar{F}C)(h) = (\bar{F}C)(0)$. For if this were not true, we could find an $n \in N$ such that $(FC^*)(n) = h_0$ with $0 < h_0 \leq h$. If δ is now defined as in the proof of Lemma 3.6, we have $\delta \in \text{Res}(L)$ and δ is increasing, so F is δ -compatible. Since $C(h) = C(0)$, we know that $C^*(x) \leq h \Leftrightarrow x \leq C(h) = C(0) \Leftrightarrow C^*(x) = 0$. In that $C^*(1_M) \leq h_1$, it is immediate that the range of $\delta \circ C^*$ is $\{0, h_1\}$. By Lemma 3.4, the range of $F(\delta \circ C^*)$ is $\{0\}$ or $\{0, h_1\}$. The fact that $F(\delta \circ C^*) = \delta \circ F(C^*)$ now shows that

$$[F(\delta \circ C^*)](n) = [\delta \circ (FC^*)](n) = \delta(h_0) = h,$$

a contradiction since $0 < h < h_1$. Thus $(\bar{F}C)(h) = (\bar{F}C)(0) = \gamma_m(r)$.

Case 3. $m < r < 1_M$. Then with δ defined as in Case 2, the range of $\delta \circ C^*$ is $\{0, h, h_1\}$, $m = C(0) = (C \circ \delta^+)(0)$, and $r = C(h) = (C \circ \delta^+)(h)$. By definition of γ_m , we now have

$$\gamma_m(r) = [\bar{F}(C \circ \delta^+)](h) = [(\bar{F}C) \circ \delta^+](h) = (\bar{F}C)(h).$$

Combining the preceding results, we are now able to state

Theorem 3.8. Let M, N be partially ordered sets with 1, each of which has more than 1 member. For a mapping $F: \text{Res}(M, L) \rightarrow \text{Res}(N, L)$, $(1) \Leftrightarrow (2) \Rightarrow (3)$. If L does not have a largest element, then also $(3) \Rightarrow (1)$.

(1) F is semiflat.

(2) F is θ -compatible for all $\theta \in \text{Res}(L)$ such that $\theta^+(0) = 0$.

(3) F is θ -compatible for all increasing $\theta \in \text{Res}(L)$.

To return to the context in which the original problem arose, we take P to be a finite set having at least 3 members, and then set $M = N = \Sigma$. By reasoning similar to that which preceded [1, Theorem 7.5], we now have the following solution to the second semiflat cluster problem.

Theorem 3.9. For an L -cluster method F , $(1) \Leftrightarrow (2) \Rightarrow (3)$. If L does not have a largest element, then also $(3) \Rightarrow (1)$.

(1) F is semiflat.

- (2) F is θ -compatible for all $\theta \in \text{Res}(L)$ such that $\theta^+(0) = 0$.
- (3) F is θ -compatible for all increasing $\theta \in \text{Res}(L)$.

4. Semiflat cluster methods in the Jardine-Sibson model

We shall phrase our discussion within the framework of Model JS, as it was outlined in [1], and we shall follow the notation of that reference. Semiflat cluster methods are defined as one would expect in Model JS. Let F be such a method with associated family $(\gamma_S)_{S \in \Sigma}$ of mappings of $\Sigma = \Sigma(P)$, the set of reflexive symmetric relations on P . We leave to the reader the routine verification of the following facts:

- (1) F satisfies JS1 $\Leftrightarrow \gamma_{(TF(d))(0)} \circ \gamma_{(Td)(0)} = \gamma_{(TF)(0)}$ for all $d \in C(P)$.
- (2) F satisfies JS2 $\Leftrightarrow (\rho \times \rho) \circ \gamma_S = \gamma_{(\rho \times \rho)(S)} \circ (\rho \times \rho)$ for all permutations ρ on P and all $S \in \Sigma$.
- (3) F necessarily satisfies JS3.
- (4) F satisfies JS4 \Leftrightarrow for each $S \in \Sigma$, the restriction of γ_S to $[S, PP]$ is increasing.
- (5) F satisfies JS5 $\Leftrightarrow S_1 \subseteq S_2$ implies $\gamma_{S_1} \leq \gamma_{S_2}$.

Suppose now that F is a cluster method in Model JS, and that F satisfies JS3 and JS5. Our goal is to show that F is semiflat if and only if it satisfies JS3⁺. Before proceeding with this, we shall need some additional terminology, as well as some preliminary results. A *directed set* is a partially ordered set D in which every finite subset has an upper bound. In a partially ordered set X , the notation $x_\delta \uparrow x$ is taken to mean that the family (x_δ) is indexed over some directed set D , that $\delta \leq \delta'$ in D implies $x_\delta \leq x_{\delta'}$ in X , and that $x = \bigvee_\delta x_\delta$ in X . Recalling that $C(P)$ is equipped with the pointwise partial order, we have

Lemma 4.1. *If $d_\delta \uparrow d$ in $C(P)$, then $F(d_\delta) \uparrow F(d)$.*

Proof. If the range of d is $\{0\}$, there is nothing to prove, so we may assume that d has a nonzero member of its range. Let k be the smallest such nonzero member. Choose ε so that $0 < \varepsilon < k$. Since $d_\delta \uparrow d$, we can find an index δ' such that $d(a, b) - d_\delta(a, b) < \varepsilon$ for all $a, b \in P$ and all $\delta \geq \delta'$. Note that $(1 - (\varepsilon/k))d \leq d_{\delta'}$, and use JS5 to establish that $F[(1 - (\varepsilon/k))d] \leq F(d_{\delta'})$. By JS3, $F[(1 - (\varepsilon/k))d] = (1 - (\varepsilon/k))F(d)$, so we have $(1 - (\varepsilon/k))F(d) \leq F(d_{\delta'})$. But $d_\delta \leq d$ implies $F(d_\delta) \leq F(d)$, so $\bigvee_\delta F(d_\delta) \leq F(d)$. We may now write

$$(1 - (\varepsilon/k))F(d) \leq F(d_\delta) \leq \bigvee_\delta F(d_\delta) \leq F(d).$$

Letting $\varepsilon \rightarrow 0$, this shows that $F(d) = \bigvee_\delta F(d_\delta)$.

We ask you now to carefully examine the proofs of Lemmas 3.3–3.7. When L is a chain that does not have a largest element (as in Model JS), what is needed

for the proofs to work is that F be θ -compatible for θ of the form

$$\theta(y) = \begin{cases} 0, & y = 0, \\ k, & 0 < y \leq h, \\ h_1, & h < y \leq h_1, \\ y, & y > h_1, \end{cases}$$

where $0 < h \leq k < h_1$. Given such a $\theta \in \text{Res}(\mathbf{R}^+)$,¹ we wish to produce a family of order automorphisms (θ_y) of \mathbf{R}^+ such that $\theta_y \uparrow \theta$. We shall consider the case where $0 < h < k < h_1$, and leave the case where $h = k$ to the reader.

For $0 < y < 1$, let $t_y = (1 - y)h$, $u_y = h + (1 - y)(h_1 - h)$ and $v_y = k + y(h_1 - k)$. Then $0 < t_y < h < u_y < h_1$, and $0 < yk < k < v_y < h_1$. Let θ_y be the order automorphism of \mathbf{R}^+ determined by the requirement that it be linear on $[0, t_y]$, $[t_y, h]$, $[h, u_y]$, $[u_y, h_1]$, and $[h_1, \infty)$ with $\theta_y(0) = 0$, $\theta_y(t_y) = yk$, $\theta_y(h) = k$, $\theta_y(u_y) = v_y$, and $\theta_y(x) = x$ for $x \geq h_1$. The graph of a typical θ_y appears in Fig. 2. Evidently $y \leq z$ implies $\theta_y \leq \theta_z$, and just as evidently, $\theta_y \uparrow \theta$.

Lemma 4.2. *If F satisfies JS3⁺, then F is θ -compatible for all θ of the form*

$$\theta(x) = \begin{cases} 0, & x = 0, \\ k, & 0 < x \leq h, \\ h_1, & h < x \leq h_1, \\ x, & x > h_1, \end{cases}$$

where $0 < h \leq k < h_1$ in \mathbf{R}^+ .

Proof. Given such a θ , we have just seen that there is a family (θ_y) of order automorphisms of \mathbf{R}^+ such that $\theta_y \uparrow \theta$. By JS3⁺, $F(\theta_y d) = \theta_y F(d)$ for all $d \in C(P)$.

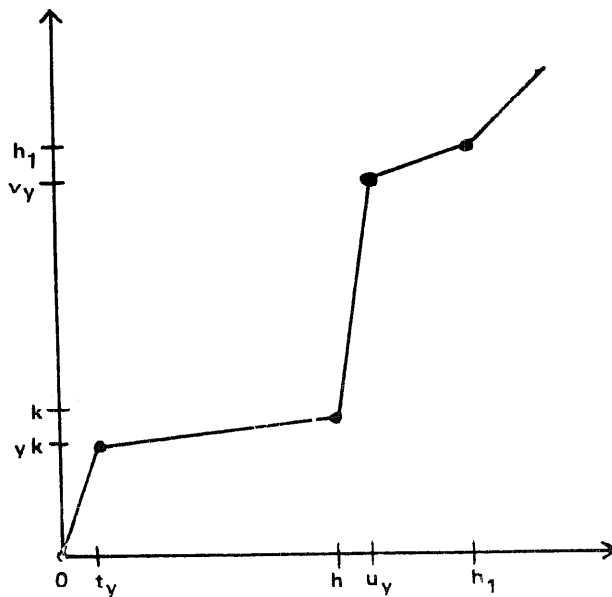


Fig. 2.

¹ Here \mathbf{R}^+ denotes the set of nonnegative real numbers equipped with the usual order.

Also $\theta_j \uparrow \theta$ implies $\theta_j d \uparrow \theta d$ and $\theta_j F(d) \uparrow \theta F(d)$ in $C(P)$. By Lemma 4.1, $F(\theta_j d) \uparrow F(\theta d)$, so $F(\theta d) = \theta F(d)$.

Theorem 4.3. *F is semiflat if and only if it satisfies JS3⁺.*

Proof. By Lemma 3.1, every semiflat cluster method satisfies JS3⁺. If a cluster method F satisfies JS3⁺ and JS5, then by Lemma 4.2, it is θ -compatible for enough residuated mappings θ for Theorem 3.8 to apply. Any such method must therefore be semiflat.

We close with a few remarks concerning the role of semiflat cluster methods within Model JS. Consider a cluster method F that satisfies JS1–JS5, JS7. Such a method is semiflat if and only if it satisfies JS3⁺; it is flat (in the sense of Jardine and Sibson) if and only if it satisfies JS3⁺ and JS8. Thus JS8 serves only to distinguish flat cluster methods from semiflat cluster methods; essentially, a semiflat cluster method satisfying JS1–JS5 is flat iff it has a closed range.

A semiflat cluster method may also be thought of as a family of flat cluster methods. One forms $S = Td(0)$, and on the basis of S , chooses a flat cluster method F_S , and takes $F(d) = F_S(d)$. When thought of in this context, it should be clear that semiflat cluster methods are frequently used, though their properties have not till now been formally studied.

References

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